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Local Currents for a Deformed Heisenberg–Poincaré Lie Algebra of Quantum Mechanics, and Anyon Statistics

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Abstract We set out to construct a Lie algebra of local currents whose space integrals, or "charges", form a subalgebra of the deformed Heisenberg–Poincaré algebra of quantum mechanics discussed by Vilela Mendes, parameterized by a fundamental length scale ℓ. One possible technique is to localize with respect to an abstract single-particle configuration space having one dimension more than the original physical space. Then in the limit $\ell \to 0$, the extra dimension becomes an unobservable, internal degree of freedom. The deformed $(1+1)$-dimensional theory entails self-adjoint representations of an infinite-dimensional Lie algebra of nonrelativistic, local currents modeled on $(2+1)$-dimensional space-time. This suggests a new possible interpretation of such representations of the local current algebra, not as describing conventional particles satisfying bosonic, fermionic, or anyonic statistics in two-space, but as describing systems obeying these statistics in a deformed one-dimensional quantum mechanics. In this context, we have an interesting comparison with earlier results of Hansson, Leinaas, and Myrheim on the dimensional reduction of anyon systems. Thus motivated, we introduce irreducible, anyonic representations of the deformed global symmetry algebra. We also compare with the technique of localizing currents with respect to the discrete positional spectrum.

Keywords Local current algebra · Deformed Lie algebra · Anyon statistics

1 Global and Local Symmetries

1.1 Heisenberg–Poincaré and Heisenberg–Euclid Symmetry Algebras

Recently Vilela Mendes and other researchers have reconsidered the combined Heisenberg and Poincaré Lie algebras as describing the kinematics of relativistic quantum mechanics

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[1, 2]. This Lie algebra is specified by the following commutation relations among the 4-vectors q_ν and p_μ ($\mu, \nu = 0, 1, 2, 3$), and the Lorentz generators $M_{\mu\nu}$: the canonical Heisenberg brackets,

$$\begin{aligned}&[p_\mu, q_\nu] = i\hbar\eta_{\mu\nu}\mathcal{J},\\&[q_\mu, q_\nu] = [p_\mu, p_\nu] = [q_\mu, \mathcal{J}] = [p_\mu, \mathcal{J}] = 0,\end{aligned} \tag{1}$$

together with the Lorentz brackets,

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\rho}\eta_{\nu\sigma}), \tag{2}$$

the brackets embodying the covariance of the 4-vectors p and q,

$$[M_{\mu\nu}, p_\lambda] = i(p_\mu\eta_{\nu\lambda} - p_\nu\eta_{\mu\lambda}), \qquad [M_{\mu\nu}, q_\lambda] = i(q_\mu\eta_{\nu\lambda} - q_\nu\eta_{\mu\lambda}), \tag{3}$$

and the final bracket expressing the centrality of the element $\mathcal{J}$,

$$[M_{\mu\nu}, \mathcal{J}] = 0,$$

where $\eta_{\mu\nu} = \text{diag}[1, -1, -1, -1]$ in units with $c = 1$.

The kinematical properties of a nonrelativistic quantum particle may be described by a self-adjoint representation in Hilbert space of the subalgebra of this Lie algebra corresponding to the spatial components only (i.e., replacing the indices μ, ν by $j, k = 1, 2, 3$). Let us call this subalgebra the Heisenberg–Euclid algebra. Then (1–3) incorporate the global translation and rotation symmetry of Minkowskian space-time, while the Heisenberg–Euclid algebra incorporates the global translation and rotation symmetry of Euclidean space.

1.2 Nonrelativistic Local Current Algebra

Let us write the *local* currents whose space integrals give us the global Heisenberg–Euclid symmetry algebra. Nonrelativistic quantum theory has been described quite generally and successfully by representing the "equal time" local current algebra of the mass density operator-valued distribution ρ and the momentum density operator-valued distribution $\mathbf{J}$, as follows [3–7]. Let f, f_1, f_2 be compactly-supported C^∞ (real-valued) scalar functions on the physical space $\mathbf{R}^3$, and let $\mathbf{g}, \mathbf{g}_1, \mathbf{g}_2$ be compactly-supported C^∞ vector fields on $\mathbf{R}^3$. Formally, one writes

$$\rho(f) = \int_{\mathbf{R}^3} \rho(\mathbf{x}) f(\mathbf{x}) d^3x, \qquad J(\mathbf{g}) = \int_{\mathbf{R}^3} J(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) d^3x, \tag{4}$$

and then

$$\begin{aligned}&[\rho(f_1), \rho(f_2)] = 0, \qquad [\rho(f), J(\mathbf{g})] = i\hbar\rho(\mathbf{g}\cdot\nabla f),\\&[J(\mathbf{g}_1), J(\mathbf{g}_2)] = -i\hbar J([\mathbf{g}_1, \mathbf{g}_2]),\end{aligned} \tag{5}$$

where $[\mathbf{g}_1, \mathbf{g}_2] = \mathbf{g}_2 \cdot \nabla\mathbf{g}_1 - \mathbf{g}_1 \cdot \nabla\mathbf{g}_2$ is the usual Lie bracket of vector fields. In the 1-particle Hilbert space $L^2_{d^3x}(\mathbf{R}^3)$, we have the self-adjoint representation

$$\begin{aligned}&[\rho(f)\Psi](\mathbf{x}) = \mathrm{m} f(\mathbf{x})\Psi(\mathbf{x}),\\&[J(\mathbf{g})\Psi](\mathbf{x}) = \frac{\hbar}{2i}\{\mathbf{g}(\mathbf{x}) \cdot \nabla\Psi(\mathbf{x}) + \nabla \cdot [\mathbf{g}(\mathbf{x})\Psi(\mathbf{x})]\},\end{aligned} \tag{6}$$

where $\Psi(\mathbf{x})$ is a square-integrable function (the probability amplitude for a single quantum particle in $\mathbf{R}^3$), and m is the particle mass. Note that the C^∞ property is needed to be able to define the bracket, and thus the infinite-dimensional Lie algebra. One may replace $\mathbf{R}^3$ by a manifold M that lacks the global symmetries, and still describe quantum mechanics by representing the local current algebra of (5).

Labeling an element of the Lie algebra by the pair $(f, \mathbf{g})$, it is useful (in anticipation of writing a deformed current algebra) to introduce (in the sense of (6)), the differential operator

$$Q(f, \mathbf{g}) = f(\mathbf{x}) + \frac{1}{2i}\{\mathbf{g}(\mathbf{x}) \cdot \nabla + \nabla \cdot \mathbf{g}(\mathbf{x})\}, \tag{7}$$

which may also be defined from (6) by $Q(f, \mathbf{g}) = (1/\mathrm{m})\rho(f) + (1/\hbar)J(\mathbf{g})$; so that $\rho(f) = \mathrm{m}Q(f, 0)$ and $J(\mathbf{g}) = \hbar Q(0, \mathbf{g})$.

Taking the test function $f(\mathbf{x})$ to approximate an indicator function $\chi_B(\mathbf{x})$ for a Borel set $B \subseteq \mathbf{R}^3$, we see that the expectation value $(\Psi, \rho(f)\Psi)$ in the 1-particle Hilbert space approximates $\mathrm{m}\int \chi_B(\mathbf{x})|\Psi(\mathbf{x})|^2 d^3x$, which is the mass times the usual probability for finding the particle in the region B. Taking $f(\mathbf{x})$ to be an approximating sequence of functions to $\delta^{(3)}(\mathbf{x} - \mathbf{x}_0)$, for a fixed point $\mathbf{x}_0 \in \mathbf{R}^3$, then $(\Psi, \rho(f)\Psi)$ approaches $\mathrm{m}|\Psi(\mathbf{x}_0)|^2$. These approximations are always to be understood in some “weak” sense—neither $\chi_B(\mathbf{x})$ nor $\delta(\mathbf{x} - \mathbf{x}_0)$ belongs to the space of compactly-supported C^∞ functions.

One easily sees how to recover the Heisenberg–Euclid algebra: if $f(\mathbf{x})$ approximates the coordinate function x_j, then $\rho(f)$ approximates the *moment* operator $\mathrm{m}q_j$ acting in $L^2_{d^3x}(\mathbf{R}^3)$ *via* multiplication by $\mathrm{m}x_j$. Similarly, if $\mathbf{g}(\mathbf{x})$ is taken to approximate a constant vector field in the j-direction, so that (let us say) $g_j(\mathbf{x}) \sim 1$ with $g_k(\mathbf{x}) = 0$ for $k \neq j$, then $J(\mathbf{g})$ approximates the differential operator $-i\hbar\partial/\partial x_j$, which is just the momentum operator p_j acting on a domain in $L^2_{d^3x}(\mathbf{R}^3)$. If I is the identity operator, we then have

$$[q_j, p_k] = i\hbar\delta_{jk}I. \tag{8}$$

Likewise the generators of spatial rotations may be recovered in the 1-particle Hilbert space. For instance, we approximate the operator generating rigid rotation about the x_3-axis (i.e., the operator for the x_3-component of orbital angular momentum) by choosing a sequence of vector fields $\mathbf{g}$, with coordinate components $g_1(\mathbf{x}) = -x_2$, $g_2(\mathbf{x}) = x_1$, and $g_3(\mathbf{x}) \equiv 0$ in the interior of a large spherical region $|\mathbf{x}| \leq R$, while outside this region we let $\mathbf{g}(\mathbf{x})$ fall smoothly to 0. Then as the radius R increases, matrix elements of $J(\mathbf{g})$ approximate matrix elements of the operator $\hbar M_{12}$ acting in $L^2_{d^3x}(\mathbf{R}^3)$ on a suitable domain.

We thus have a clear relation in $\mathbf{R}^3$ between the usual 1-particle, self-adjoint representation of the local current algebra, and an irreducible representation of the global symmetry algebra. Similar statements hold for $\mathbf{R}^d$, $d = 1, 2, \ldots$.

The infinite-dimensional *group* obtained by exponentiating the local current commutators is the natural semidirect product of the group of compactly-supported C^∞ scalar functions with the group of compactly-supported diffeomorphisms of the spatial manifold.

1.3 Bose, Fermi, and Intermediate Quantum Statistics

The local current algebra has many other, mutually inequivalent, irreducible self-adjoint representations, which can be obtained and understood by classifying the continuous irreducible unitary representations of the corresponding group [5, 7]. In particular, for space dimension $d \geq 2$, we have for each natural number $N \geq 2$ unitarily inequivalent representations that describe N indistinguishable particles obeying Bose or Fermi statistics. These may be obtained *via* inducing from the 1-dimensional unitary representations of the symmetric group S_N [8]. For $N \geq 3$, we also have inequivalent representations induced by the higher-dimensional representations of S_N, describing particles obeying *parastatistics* [7, 9].

Moreover, in two space dimensions we have representations that describe N indistinguishable particles obeying the intermediate statistics of *anyons* [10–16], induced by the 1-dimensional unitary representations of the *braid group* B_N [6, 14, 17]. Here a single counterclockwise exchange of two particles multiplies the wave function by a phase $\exp[i\pi\lambda]$, where (for each such representation) λ is fixed between 0 and 2. The value $\lambda = 1$ corresponds to fermions, while $\lambda = 0$ corresponds to bosons. In addition, there exist representations in two-space (for $N \geq 3$) induced by higher-dimensional braid group representations (describing particles called plektons) [18].

The case of one space dimension is still different. Here, the Bose and Fermi and intermediate representations of the local current algebra are all unitarily equivalent. The different possibilities for quantum statistics, including the intermediate statistics, are associated with a one-parameter family of distinct self-adjoint extensions of the differential operator for the kinetic energy part of the Hamiltonian [5, 10]. These in turn are characterized by a boundary condition on wave functions in the domain of the Hamiltonian operator, which applies where the coordinate separation of adjacent particles is zero: $(\partial\Psi/\partial x)|_{x=0} = \eta\Psi|_{x=0}$, where $x > 0$ is the relative coordinate between adjacent particles in one-dimensional space.

Hansson, Leinaas, and Myrheim have considered two ways to relate intermediate statistics in two-dimensional space with intermediate statistics in one dimension [19]. One is to confine anyonic particles in two-space by means of a narrow potential well. The second is to consider anyons moving in a strong magnetic field, restricting attention to the lowest Landau level. The present article offers a third possibility: taking the $\ell \to 0$ limit of anyonic representations of an ℓ-deformed algebra of currents that we shall discuss in the next section.

In each of the aforementioned representations of the nonrelativistic current algebra, for arbitrary finite N, it is possible to obtain from the local currents a representation of the Heisenberg–Euclid algebra as a global symmetry algebra. As before, one approximates the moment operators in each spatial direction, the total momentum operators in each spatial direction, and the total orbital angular momentum operators about each axis. However, the representations of the global symmetry algebra thus obtained are, in general, no longer irreducible. They are tensor products (symmetric, antisymmetric, or braided) of irreducible representations [20].

1.4 A Deformed Lie Algebra for Quantum Mechanics

Now the Heisenberg–Poincaré algebra of (1–4) has nontrivial second cohomology, a necessary condition for it to be deformable [21]. The members of a family of nontrivial deformations of this Lie algebra, parameterized by two length parameters ℓ and R, have the property of "stability" or "rigidity"—meaning that small changes in their structure constants produce isomorphic Lie algebras [22–25]. This deformation by ℓ and R, discussed by Vilela Mendes, is a particular choice among other possibilities and was the starting point for the present work. Chryssomalakos and Okon discuss all the possible stable deformations of the Heisenberg–Poincaré algebra, including an explanation of the relevant cohomology theory and detailed references.

In the deformed Heisenberg–Poincaré algebra, (2–4) are unchanged, while (1) are replaced by the following:

$$
\begin{aligned}
&[p_\mu, q_\nu] = i\hbar\eta_{\mu\nu}\mathcal{J}, \qquad [q_\mu, q_\nu] = -i\varepsilon\ell^2 M_{\mu\nu},\\
&[p_\mu, p_\nu] = -i\frac{\varepsilon'\hbar^2}{R^2}M_{\mu\nu}, \qquad [q_\mu, \mathcal{J}] = i\varepsilon\frac{\ell^2}{\hbar}p_\mu, \qquad [p_\mu, \mathcal{J}] = -i\frac{\varepsilon'\hbar}{R^2}q_\mu,
\end{aligned}
\tag{9}
$$

where ε and ε' are ± 1. As $\ell \to 0$ and $R \to \infty$, we recover the original algebra. This deformed Lie algebra is isomorphic to the Lie algebra of the orthogonal group in six dimensions, with metric $\eta_{ab} = \mathrm{diag}[1, -1, -1, -1, \varepsilon', \varepsilon]$. The limit procedure here is already anticipated in the method of reduction of certain representations of $SO(4, 2)$ by Barut and Bohm [26]. In our nonrelativistic framework, it is natural to study the case $\varepsilon = -i$, where time has a continuous spectrum.

Since the (small) parameter ℓ is relevant locally, we follow Vilela Mendes in focusing on the algebra obtained by taking $R \to \infty$, so that the right-hand sides of the brackets involving R in (9) become zero. To describe the corresponding nonrelativistic quantum mechanics, we consider self-adjoint representations of the resulting modified Heisenberg–Euclid subalgebra, given by the spatial components of (2–4) together with the brackets:

$$
\begin{aligned}
&[q_j, q_k] = i\ell^2 M_{jk}, \qquad [q_j, p_k] = i\delta_{jk}\hbar\mathcal{J}, \qquad [q_j, \mathcal{J}] = -i\frac{\ell^2}{\hbar}p_j, \\
&[p_j, p_k] = [p_j, \mathcal{J}] = 0,
\end{aligned}
\tag{10}
$$

where $j, k = 1, 2, 3$ (and $\varepsilon = -1$). The Lie algebra of (2–4) for $j, k = 1, 2, 3$, together with (10), now describes the *global* symmetry of the deformed quantum theory. An especially interesting feature is that in a self-adjoint representation of this deformed symmetry algebra, the coordinate operators q_μ no longer commute.

1.5 The Problem of Local Current Algebra for the Deformed Symmetry

We thus arrive at the problem situation described by the four cells of Fig. 1. The two cells in the upper row refer to *finite-dimensional* Lie algebras describing global symmetry; those in the lower row refer to *infinite-dimensional* Lie algebras of local currents describing local symmetry. The two cells in the left-hand column refer to the *undeformed* Lie algebras describing the symmetry of conventional nonrelativistic quantum mechanics; those in the right-hand column refer to the Lie algebras of the *deformed* theory with $\ell > 0$.

	undeformed Lie algebras		deformed Lie algebras
global symmetry	Heisenberg-Euclid algebra of quantum mechanics	$\Leftarrow$ $\lim_{\ell\to 0}$ $\Leftarrow$	spatial subalgebra of the ℓ-deformed Heisenberg-Poincaré Lie algebra
	$\Uparrow$ $\int_{\text{all space}} \cdots d\mathbf{x}$ $\Uparrow$		$\Uparrow$ $\int_{\text{all space}} \cdots d\mathbf{x}$ $\Uparrow$
local symmetry	nonrelativistic local current algebra of scalar functions and vector fields	$\Leftarrow$ $\lim_{\ell\to 0}$ $\Leftarrow$	? ℓ-deformed local current algebra (to be constructed)

Fig. 1 Desired characteristics of the deformed local current algebra

Evidently our goal is to construct (and represent by self-adjoint operators in Hilbert space) an infinite-dimensional local current algebra (in the cell on the lower right), parameterized by ℓ, with the following properties. When we average the local currents with test functions that (in an appropriate weak limit) become constant across all of space, we should recover the subalgebra of (2–4) together with (10), i.e., we move from the lower right-hand cell to the upper right-hand cell. Moreover, when we take ℓ to zero, we should recover (in some mathematical sense) the usual local current algebra of nonrelativistic quantum mechanics, moving from the cell on the lower right to the cell on the lower left.

In addition, we shall need to pay attention to the Hamiltonian operator in such an ℓ-deformed theory, and what happens to it as $\ell \to 0$.

2 Deformed Local Current Algebras

In Ref. [27], we developed and discussed in some detail two alternative ways to construct an infinite-dimensional Lie algebra meeting these specifications. Each has certain advantages and disadvantages. In this section, we briefly outline some of those results and expand on the discussion, focusing on the case where the spatial dimension $d = 1$.

2.1 Local Current Algebra in Augmented Physical Space

First let us rewrite (6) for a two-dimensional Euclidean space, with coordinates (x, w). This will provide a way to represent local currents for the Lie algebra of (10) with $d = 1$ (so that the indices $j = k = 1$). We think of x as the usual spatial coordinate for the particle moving in one dimension, and we regard w as extending the spatial manifold by an additional dimension that is unobserved in conventional physics.

We have then a class of self-adjoint operators $Q(h, g_x, g_w)$ in $L^2_{dxdw}(\mathbf{R}^2)$, where $h(x, w)$ is a compactly-supported, real-valued C^∞ function on $\mathbf{R}^2$, and where g_x, g_w are the coordinate components of a compactly-supported, C^∞ vector field on the xw-plane:

$$[Q(h, g_x, g_w)\Psi](x, w) = \left\{h(x, w) + \frac{1}{2i}\left[g_x(x, w)\frac{\partial}{\partial x} + \frac{\partial}{\partial x}g_x(x, w)\right] + \frac{1}{2i}\left[g_w(x, w)\frac{\partial}{\partial w} + \frac{\partial}{\partial w}g_w(x, w)\right]\right\}\Psi(x, w). \quad (11)$$

We recover (6) in two space dimensions by setting $\rho(h) = \mathrm{m}Q(h, 0, 0)$, and $J(\mathbf{g}) = \hbar Q(0, g_x, g_w)$. Now let $\ell > 0$ be a small length parameter, and define

$$Q_\ell(h, g_x, g_w) = Q(h, \ell g_x, \ell g_w). \quad (12)$$

Then we actually have in (11–12) a parameterized family of Lie algebras represented by self-adjoint operators.

Suppose we fix $\ell \neq 0$. Let $h(x, w)$ approach the coordinate function x, and let $(g_x(x, w), g_w(x, w))$ approach the vector field with coordinate components $(-w, x)$. Then we recover from Q_ℓ the operator q, given by

$$q = x + i\ell\left(w\frac{\partial}{\partial x} - x\frac{\partial}{\partial w}\right). \quad (13)$$

Let us further set $p = (\hbar/\ell)\lim_{g_x\to 1} Q_\ell(0, g_x, 0)$ (the limit being taken as g_x approaches the constant component function of magnitude 1), and $\mathcal{J} = \lim_{h\to 1, g_w\to 1} Q_\ell(h, 0, g_w)$ (the limit taken as both h and g_w become identically 1). Then we have

$$p = -i\hbar\frac{\partial}{\partial x}, \qquad \mathcal{J} = I - i\ell\frac{\partial}{\partial w}. \tag{14}$$

The operators of (13) and (14) represent the deformed Heisenberg brackets with $d = 1$, as desired:

$$[q, p] = i\hbar\mathcal{J}, \qquad [q, \mathcal{J}] = -i\frac{\ell^2}{\hbar}p, \qquad [p, \mathcal{J}] = 0. \tag{15}$$

Equations (13–14) actually form a *reducible* representation of the Lie algebra (15), in the Hilbert space $L^2_{dxdw}(\mathbf{R}^2)$. This might be seen as a disadvantage to our approach. However, we already have the experience of obtaining reducible representations of the global symmetry algebra from irreducible representations of the local current algebra (in the case of N indistinguishable particles, $N > 1$, as described in Sect. 1). Thus we do not interpret this as a fundamental objection.

A local current corresponding to p is

$$J(g) = \frac{\hbar}{2i}\left\{g(x)\frac{\partial}{\partial x} + \frac{\partial}{\partial x}g(x)\right\}, \tag{16}$$

where $g(x)$ is a compactly-supported C^∞ function on $\mathbf{R}$, while a local current corresponding to $\mathcal{J}$ is

$$\mathcal{J}(k) = k(w) + \frac{\ell}{2i}\left\{k(w)\frac{\partial}{\partial w} + \frac{\partial}{\partial w}k(w)\right\}, \tag{17}$$

where $k(w)$ is likewise a compactly-supported C^∞ function on $\mathbf{R}$. These currents embody the (infinitesimal generators of) compactly-supported flows in the separate coordinate directions x, w.

But it does not suffice for our purposes to use local currents depending separately on x and w, because of the way the operator q mixes the x and the w directions. This is why we must incorporate dependence on both variables in the test functions h, g_x and g_w that appear as arguments of Q_ℓ, and use the full Lie algebra of local currents in two dimensions. The local currents in (16–17) are expressed in terms of Q_ℓ by the equations,

$$J(g) = \lim_{g_x\to g}\frac{\hbar}{\ell}Q_\ell(0, g_x, 0), \qquad \mathcal{J}(k) = \lim_{h\to k}\lim_{g_w\to k}Q_\ell(h, 0, g_w), \tag{18}$$

where the (weak) limit of functions refers to the fact that the test functions g and k depend only on x, whereas the test functions h, g_x, g_w (being compactly-supported in the xw-plane) must depend on both variables.

In the representation of the global symmetry algebra by (13–14), the operators approach the standard representation of the Heisenberg algebra as $\ell \to 0$. Note that in this limit the w coordinate is still present, but it becomes unobservable (in our physical interpretation).

Finally, we consider how to obtain the local current algebra for one space dimension, when $\ell \to 0$. The operator $Q_\ell(h, g_x, g_w)$ becomes the multiplication operator $Q(h, 0, 0)$ for any choice of g_x, g_w. Letting $f(x)$ be a real-valued, compactly-supported C^∞ function on the line, we thus have the mass density operator

$$\rho(f) = \lim_{h\to f}\mathrm{m}Q(h, 0, 0) = \lim_{\ell\to 0}\lim_{h\to f}\mathrm{m}Q_\ell(h, g_x, g_w). \tag{19}$$

But recovery of the momentum density operators $J(g)$ from the representations Q_ℓ in the $\ell \to 0$ limit is slightly more problematic. One alternative is to introduce $J(g)$ anew in this limit, writing

$$J(g) = \lim_{g_x \to g} \lim_{\ell \to 0} \frac{\hbar}{\ell} Q_\ell(0, g_x, 0). \tag{20}$$

But since we already have the desired expression for $J(g)$ from (18), before taking $\ell \to 0$, another possibility is to take the combined Lie algebra represented by the operators $Q_\ell(h, g_x, g_w)$ [with test functions compactly-supported in $\mathbf{R}^2$], *together* with the operators $J(g)$ [with test functions compactly-supported in $\mathbf{R}^1$], as the desired ℓ-deformed local current algebra.

In short, referring to the second column of Fig. 1, the Lie algebra of (15) expresses the *global* symmetry of the deformed quantum theory in one space dimension. One way to obtain a description of a corresponding *local* symmetry is by (11) and (12), based on the usual local current algebra on a space augmented by the coordinate w, together with (16).

Referring to the first row of Fig. 1, we have that (13), (14), and (15) all go over smoothly (as $\ell \to 0$) to the Heisenberg algebra in one dimension and the usual operators representing it (albeit in a reducible representation that retains the variable w). Referring to the second row of Fig.1, in the $\ell \to 0$ limit, we recover the usual, irreducible representation of the local current algebra in one space dimension from an irreducible representation of the ℓ-deformed local current algebra described here.

In this subsection, we have elaborated considerably on the discussion in Ref. [27]. In that article we also discuss various unitarily equivalent representations of the global symmetry algebra, the direct integral decomposition of reducible representations into irreducibles, and the sense in which the Heisenberg algebra is obtained as the $\ell \to 0$ limit of a family of irreducible representation of the global symmetry algebra.

To write irreducible representations of (15), note that the Casimir operator

$$C = \frac{1}{\hbar^2} p^2 + \frac{1}{\ell^2} \mathcal{J}^2 \tag{21}$$

commutes with the generators q, p, and $\mathcal{J}$. In an irreducible representation, C takes the value ρ_0^2. The self-adjoint operator representing q in such an irreducible representation has a discrete eigenvalue spectrum, given by $\{n\ell, n \in \mathbf{Z}\}$. Hence another alternative, which we next describe, is to try to work within a single irreducible representation of the global symmetry algebra, interpreting "locality" with respect to this positional spectrum.

2.2 A Discretized Infinite-Dimensional Algebra

In the Hilbert space spanned by the complete orthonormal set of vectors $\{|n\ell\rangle, n \in \mathbf{Z}\}$, with $q|n\ell\rangle = n\ell|n\ell\rangle$, we have

$$\begin{aligned} &\langle n\ell|q|m\ell\rangle = \delta_{n,m} m\ell, \qquad \langle n\ell|p|m\ell\rangle = \frac{\hbar\rho_0}{2i}(\delta_{n+1,m} - \delta_{n-1,m}), \\ &\langle n\ell|\mathcal{J}|m\ell\rangle = \frac{\ell\rho_0}{2}(\delta_{n+1,m} + \delta_{n-1,m}), \end{aligned} \tag{22}$$

and $C = \rho_0^2 I$. Since

$$q = \sum_{n=-\infty}^{\infty} n\ell |n\ell\rangle\langle n\ell|, \tag{23}$$

the corresponding *local* mass density operator J_q should be defined by

$$\mathrm{J}_q(g) = \mathrm{m} \sum_{n=-\infty}^{\infty} g(n\ell)|n\ell\rangle\langle n\ell|, \tag{24}$$

where the *compactly-supported* real-valued functions g are such that for all but finitely many $n \in \mathbf{Z}$, $g(n\ell) = 0$. Then, as $g(n\ell)$ (weakly) approximates the function $n\ell$, $\mathrm{J}_q(g)$ approximates the moment operator $\mathrm{m}q$. Note that when $g(n\ell)$ approximates the constant function 1, $\mathrm{J}_q(g)$ approximates $\mathrm{m}I$, where I is the identity operator.

To write local currents corresponding to the operators p and $\mathcal{J}$, define

$$\mathrm{J}_p(h) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{h}(n\ell)\{p|n\ell\rangle\langle n\ell| + |n\ell\rangle\langle n\ell|p\}, \tag{25}$$

$$\mathrm{J}_{\mathcal{J}}(r) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{r}(n\ell)\{\mathcal{J}|n\ell\rangle\langle n\ell| + |n\ell\rangle\langle n\ell|\mathcal{J}\}, \tag{26}$$

where $h(n\ell) \equiv (1/2)[\tilde{h}(n\ell) + \tilde{h}((n+1)\ell)]$ and $r(n\ell) \equiv (1/2)[\tilde{r}(n\ell) + \tilde{r}((n+1)\ell)]$ are also compactly supported. Equivalently,

$$\mathrm{J}_p(h) = \frac{\hbar\rho_0}{2i} \sum_{n=-\infty}^{\infty} h(n\ell)\{|n\ell\rangle\langle(n+1)\ell| - |(n+1)\ell\rangle\langle n\ell|\}, \tag{27}$$

$$\mathrm{J}_{\mathcal{J}}(r) = \frac{\ell\rho_0}{2} \sum_{n=-\infty}^{\infty} r(n\ell)\{|n\ell\rangle\langle(n+1)\ell| + |(n+1)\ell\rangle\langle n\ell|\}. \tag{28}$$

When $\tilde{h}(n\ell)$ and $\tilde{r}(n\ell)$ become identically 1, so do $h(n\ell)$ and $r(n\ell)$. Then $\mathrm{J}_p(h)$ approximates p, $\mathrm{J}_{\mathcal{J}}(r)$ approximates $\mathcal{J}$, and the global symmetry algebra is recovered.

Thus the infinite-dimensional Lie algebra generated by $\mathrm{J}_q(g)$, $\mathrm{J}_p(h)$, and $\mathrm{J}_{\mathcal{J}}(r)$ is another candidate for the lower right-hand entry in Fig. 1. It integrates to the correct global symmetry algebra, and may be written in the Hilbert space of a single irreducible representation of the global algebra. Moreover, the localization of the currents has a more immediate physical interpretation, without the introduction of an additional spatial dimension.

However, the limit $\ell \to 0$ is problematic. To recover the usual irreducible representation of Heisenberg algebra from the $\ell \to 0$ limit of irreducible representations of the deformed global algebra, we showed in Ref. [27] that we needed to simultaneously take $\rho_0 \to \infty$, with $\rho_0 \sim 1/\ell$. But this does not carry over to the corresponding local algebras.

In order that the Lie algebra generated by the discretized currents be *local*, in the irreducible representation labeled by ρ_0, we would need the commutator brackets of the operators $\mathrm{J}_q(g)$, $\mathrm{J}_p(h)$, and $\mathrm{J}_{\mathcal{J}}(r)$ given by (24), (27), and (28) to yield similarly local expressions. These expressions are all linear combinations of operators of the form $|n\ell\rangle\langle n\ell|$, $|n\ell\rangle\langle(n+1)\ell|$, and $|(n+1)\ell\rangle\langle n\ell|$. We straightforwardly obtain

$$[\mathrm{J}_q(g_1), \mathrm{J}_q(g_2)] = 0, \tag{29}$$

$$[\mathrm{J}_q(g), \mathrm{J}_p(h)] = -i\frac{\mathrm{m}\hbar}{\ell}\mathrm{J}_{\mathcal{J}}(r), \tag{30}$$

where $r(n\ell) = h(n\ell)\{g(n\ell) - g([n+1]\ell)\}$, and

$$[\mathrm{J}_q(g), \mathrm{J}_{\mathcal{J}}(r)] = i\frac{\mathrm{m}\ell}{\hbar}\mathrm{J}_p(h), \tag{31}$$

where $h(n\ell) = r(n\ell)(g(n\ell) - g([n+1]\ell))$, which thus far are satisfactorily local. But other commutators, such as $[\mathrm{J}_p(h), \mathrm{J}_{\mathcal{J}}(r)]$, generate terms with $|(n+1)\ell\rangle\langle(n-1)\ell|$ and $|(n-1)\ell\rangle\langle(n+1)\ell|$. Successive commutators generate additional terms $|(n-m)\ell\rangle\langle(n+m)\ell|$ and $|(n+m)\ell\rangle\langle(n-m)\ell|$, for arbitrary $m \in \mathbf{Z}$. To close the Lie algebra of these currents, one is thus forced to include new basis elements in the current algebra, having more general forms; e.g.,

$$\sum_{n,m=-\infty}^{\infty} s(n\ell, m\ell)\{|(n+m)\ell\rangle\langle(n-m)\ell| \pm |(n-m)\ell\rangle\langle(n+m)\ell|\},$$

where s is compactly supported on the square lattice of points $(n\ell, m\ell)$. Such currents are *nonlocal* in the positional eigenvalues, since $(n-m)\ell$ and $(n+m)\ell$ become arbitrarily far apart. This sort of behavior by the commutation relations of discretized local derivatives is well-known in the context of lattice models. But in our context, it poses additional difficulties in recovering the nonrelativistic local current algebra in the $\ell \to 0$ limit.

We also note that a theory of space-time based on the discrete positional spectrum of the operator q moves in the direction of the finite-dimensional quantum theory advocated by David Finkelstein, in his January 2006 lecture in Oberwolfach [28].

2.3 The Kinetic Energy Hamiltonian

In [1] it was suggested that for a particle of mass m, we should use for the kinetic energy Hamiltonian the operator $H_0 = p^2/2\mathrm{m}$, where p is the generator appearing in the algebra of (15), and that the oscillator Hamiltonian should then be $H_{\mathrm{osc}} = p^2/2\mathrm{m} + \mathrm{m}\omega^2 q^2/2$. But we argue for a different choice—H_0 should lead to the physical, kinematical condition that the time-derivative of the particle position is the particle velocity. That is, we should expect H_0 (as well as H_{osc}) to satisfy,

$$\dot{q} = \frac{1}{i\hbar}[q, H_0] = \frac{1}{i\hbar}[q, H_{\mathrm{osc}}] = \frac{p}{\mathrm{m}}. \tag{32}$$

But from the deformed Heisenberg brackets in (15), we have the fact that

$$(1/i\hbar)[q, p^2/2\mathrm{m}] = (p\mathcal{J} + \mathcal{J}p)/2\mathrm{m}. \tag{33}$$

The right-hand side of (33) equals p/m when $\mathcal{J}$ is the identity operator, but otherwise it does not! To fulfill (32), we change the form of the kinetic energy term in the Hamiltonian, setting

$$H_0 = \frac{1}{2\mathrm{m}}\left\{p^2 + \frac{\hbar^2}{\ell^2}(\mathcal{J} - I)^2\right\}, \tag{34}$$

where I is the identity operator in the representation of the Lie algebra.

The coefficient $\hbar^2/\ell^2$ in (34) is *required* in order to obtain the correct bracket with q. In the representation of (13–14), we then have

$$H_0 = -\frac{\hbar^2}{2\mathrm{m}}\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial w^2}\right\}. \tag{35}$$

Note that this form does *not* change when $\ell \to 0$. Our interpretation is that wave functions in low-energy states become decreasingly dependent on the variable w as $\ell \to 0$, so that the second term of (35) tends to vanish for observable particle states in this limit. The above choice for H_0 also opens the way to multiparticle states satisfying anyonic statistics, as introduced in the following section.

Note that taking the full Hamiltonian H to be $H_0 + V(q)$, we continue to have $\dot{q} = p/\mathrm{m}$. However, in the representation of (13–14), the operator $V(q)$ is not a multiplication operator but in general some higher-order differential operator.

Let us conclude with a brief discussion of equations of continuity for the deformed quantum theory. With Q_ℓ acting *via* (11–12) in $L^2_{dxdw}(\mathbf{R}^2)$, we already have operators $\rho_\ell(h) = \mathrm{m}Q_\ell(h,0,0)$, $J_\ell(\mathbf{g}) = (\hbar/\ell)Q_\ell(0, g_x, g_w)$, and the free Hamiltonian $H_{0\ell}$ given by (35). Just as in the standard quantum mechanics for two space dimensions, we then have

$$\dot{\rho}_\ell(f) = \frac{1}{i\hbar}[\rho_\ell(f), H_{0\ell}] = J_\ell(\nabla f), \tag{36}$$

which is a kind of continuity equation for the mass density in the augmented space (i.e., in xw-space). However, when $\ell \neq 0$, this does not have the interpretation of an equation of continuity for the mass density in *positional* space. Furthermore, it no longer holds if H_0 is replaced by $H = H_0 + V(q)$.

But within the framework of the discretized current algebra, we have an equation of continuity relating the time-derivative of J_q to the spatial divergence of J_p. Taking the Hamiltonian H to be $H_0 + V(q)$, with H_0 given by (34), one obtains after a rather lengthy calculation,

$$\dot{\mathrm{J}}_q(g) = \frac{1}{i\hbar}[\mathrm{J}_q(g), H] = \mathrm{J}_p(Dg), \tag{37}$$

where

$$Dg(n\ell) \equiv \frac{g((n+1)\ell) - g(n\ell)}{\ell} \tag{38}$$

is the discretized derivative. As noted above, the density J_q and current J_p in this continuity equation are local, but belong to a Lie algebra that necessarily includes nonlocal operators.

3 Anyonic Representations

In this section we sketch briefly how anyonic representations arise in the present framework. Further details will be provided in subsequent work.

First consider the local current we discussed in Sect. 2.1; i.e., the combined Lie algebra represented by $Q_\ell(h, g_x, g_w)$ [with test functions having compact support in $\mathbf{R}^2$] and $J(g)$ [with test functions compactly-supported in $\mathbf{R}^1$]. A 2-particle representation of this local current algebra, where the particles are indistinguishable, acts in the Hilbert space of wave functions on the configuration space of 2-point subsets of xw-space, square-integrable with respect to a (local) Lebesgue measure. Let us write such a wave function as $\Psi^{(2)}(x_1, w_1; x_2, w_2)$, with the coordinates indexed so that $x_1 < x_2$, or when $x_1 = x_2$, we have $w_1 < w_2$. Then we have the representation

$$\begin{aligned}Q_\ell(h,g_x,g_w)\Psi^{(2)} = \Big\{ & h(x_1,w_1) + \frac{\ell}{2i}\left[g_x(x_1,w_1)\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1}g_x(x_1,w_1)\right] \\ & + \frac{\ell}{2i}\left[g_w(x_1,w_1)\frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_1}g_w(x_1,w_1)\right]\Big\}\Psi^{(2)} \\ & + \Big\{h(x_2,w_2) + \frac{\ell}{2i}\left[g_x(x_2,w_2)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_2}g_x(x_2,w_2)\right] \\ & + \frac{\ell}{2i}\left[g_w(x_2,w_2)\frac{\partial}{\partial w_2} + \frac{\partial}{\partial w_2}g_w(x_2,w_2)\right]\Big\}\Psi^{(2)}, \end{aligned} \tag{39}$$

along with

$$J(g)\Psi^{(2)} = \frac{\hbar}{2i}\left\{g(x_1)\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1}g(x_1)\right\}\Psi^{(2)} + \frac{\hbar}{2i}\left\{g(x_2)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_2}g(x_2)\right\}\Psi^{(2)}, \tag{40}$$

where the final arguments in the sense of (11) are suppressed. Since the test function g depends on x but not on w, the operator $J(g)$ generates independent translations of the two particles in the x-direction. Such translations do not alter the order relation $x_1 < x_2$, or $x_1 = x_2$. On the other hand, the operator $Q_\ell(0, g_x, g_w)$ generates general, independent motions of the points at (x_1, w_1) and (x_2, w_2). Thus we have the possibility, as in Ref. [11], of generating a rotation of one particle about the other. One way to obtain anyon statistics is in the selection of a corresponding boundary condition on the two-particle wave functions. That is, rewrite $\Psi^{(2)}$ using the relative coordinates $(x, w) = (x_2 - x_1, w_2 - w_1)$ and, say, the center of position coordinates $(x_0, w_0) = (1/2)(x_2 + x_1, w_2 + w_1)$. Express (x, w) in polar coordinates (r, θ) with $r > 0, -\pi/2 \le \theta \le \pi/2$. Then we may impose an "anyonic" boundary condition, $\Psi^{(2)}|_{\theta=\pi/2} = \exp[i\pi\lambda]\Psi^{(2)}_{\theta=-\pi/2}$, on wave functions in the domain of the local current operators.

As noted earlier, the Hilbert space for an irreducible representation of the local current algebra may be expressed as a direct integral of irreducible representations of the global symmetry algebra. As described in Refs. [1] and [27], an irreducible representation in which the Casimir operator C acts as $\rho_0^2 I$ may be written in the Hilbert space $\widehat{\mathcal{H}}_{\rho_0}$ of square-integrable functions on a circle of radius ρ_0 in the $k_x k_w$-plane, where (k_x, k_w) are the coordinates obtained by Fourier transform of a representation in xw-space. Introducing polar coordinates (ρ, ψ) with $k_x = \rho \sin\psi, k_w = \rho\cos\psi, dk_x dk_w = \rho d\rho d\psi$, the operators become

$$\begin{aligned}\hat{q} &= i\ell\frac{\partial}{\partial\psi}, \\ \hat{p} &= \hbar\rho\sin\psi, \\ \widehat{\mathcal{J}} &= \ell\rho\cos\psi.\end{aligned} \tag{41}$$

Now $\hat{p}$ and $\widehat{\mathcal{J}}$ are bounded operators, defined on all of $\widehat{\mathcal{H}}_{\rho_0}$, while $\hat{q}$ is symmetric but unbounded—thus its domain of definition must be specified. The usual eigenstates of $i\ell\partial/\partial\psi$ in $\widehat{\mathcal{H}}_{\rho_0}$ are given by wave functions $\widehat{\Psi}_n(\psi) = \exp[-in\psi]$, so that $\hat{q}\widehat{\Psi}_n = n\ell\widehat{\Psi}_n$, justifying (22–23). These eigenfunctions and their linear combinations satisfy $\widehat{\Psi}|_{\psi=2\pi} = \widehat{\Psi}|_{\psi=0}$, a periodic boundary condition on elements of the domain of $\hat{q}$ that is unchanged by the multiplication operators $\hat{p}$ and $\widehat{\mathcal{J}}$. However, there is a one-parameter family $\hat{q}^{(\alpha)}$ of distinct self-adjoint operators, described by the differential operator $i\ell\partial/\partial\psi$ acting in $\widehat{\mathcal{H}}_{\rho_0}$ on domains of wave functions obeying boundary conditions $\widehat{\Psi}^{(\alpha)}|_{\psi=2\pi} = \exp[-2i\pi\alpha]\widehat{\Psi}^{(\alpha)}|_{\psi=0}$

($0 \le \alpha < 1$). The eigenfunctions of $\hat{q}^{(\alpha)}$ are $\widehat{\Psi}_n^{(\alpha)}(\psi) = \exp[-i(n+\alpha)\psi]$, with corresponding eigenvalues $(n+\alpha)\ell$. Again, the boundary condition is unchanged by the multiplication operators $\hat{p}$ and $\widehat{\mathcal{J}}$. Since the spectrum of the operator representing q is different in each representation, they cannot be unitarily equivalent. Of course, for a single free particle, we may choose to remove α by (arbitrarily) changing the origin with respect to which "position" is defined—in effect, redefining the position operator in such a representation to be $\hat{q}^{(\alpha)} - \alpha\ell I$. But in the two-particle case, redefining the origin cannot change the *relative* coordinate. Here, the $\alpha \neq 0$ representations of the global symmetry algebra lead us again to anyon statistics.

We showed in Ref. [27] that the appropriate way to recover the Heisenberg algebra from irreducible representations of the deformed global symmetry algebra is to let $\rho_0 \to \infty$ as $\ell \to 0$, with $\rho_0 \sim 1/\ell$. Our final remark is to suggest the possibility that the boundary condition $(\partial\Psi/\partial x)|_{x=0} = \eta\Psi|_{x=0}$ on wave functions in the domain of the kinetic energy Hamiltonian can result from the process of taking the $\ell \to 0$ limit of N-anyon representations of the deformed local current algebra, providing an interesting new context for dimensional reduction for anyon systems in the spirit of earlier ones proposed by Hansson, Leinaas, and Myrheim.

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